Linear stability

Solving problems : $\frac{d\underline{x}}{dt} = \underline{f}(\underline{x}, \underline{u})$

which when linearized can be written as

$$\boxed{\begin{gathered}\frac{d\Delta x}{dt} = A\Delta x + B\Delta u \\ \underline{x} = \underline{x_e} + \Delta x\end{gathered}}$$

Stability: If system starts near an equilibrium point, does it converge to or move away from equilibrium.

How to determine stability?

Ex | Single Input system

$y = y_e + \Delta y$ $\qquad \Delta y(t=0) = \Delta y_0$

$\frac{dy}{dt} = \alpha \Delta y$ $\qquad \Rightarrow \Delta y = e^{\alpha t} \Delta y_0$

$$\boxed{y = y_e + e^{\alpha t} \Delta y_0}$$

Let $\alpha = \sigma + j\omega$

$y = y_e + (e^{\sigma t} \Delta y_0) e^{j\omega t}$ $\qquad e^{j\omega t} \Rightarrow$ oscillations

$\sigma > 0$: exponential growth $\Rightarrow$ unstable

$\sigma < 0$: exponential decay $\Rightarrow$ stable

$\sigma = 0$: pure oscillations $\Rightarrow$ marginally stable.

* What about multiple input systems?

Ex] $\frac{d\underline{x}}{dt} = \underline{f}(\underline{x})$

$\underline{x} = \underline{x}_e + \underline{\Delta x}$ $\qquad$ $\underline{\Delta x}(t=0) = \underline{\Delta x}_0$

$\frac{d\underline{\Delta x}}{dt} = A\underline{\Delta x}$ $\qquad$ $\Rightarrow \underline{\Delta x} = \exp(At)\underline{\Delta x}_0$

$$\exp(At) = I + At + \frac{1}{2}(At)^2 + \ldots.$$

Recall: $A = V\Lambda V^{-1} \Rightarrow \exp(At) = I + V\Lambda V^{-1}t + \frac{1}{2}(V\Lambda V^{-1}t)^2 + \ldots.$

$$= VIV^{-1} + V\Lambda tV^{-1} + \frac{1}{2}(V\Lambda tV^{-1})^2 + \ldots.$$

$$= V\left(I + \Lambda t + \frac{1}{2}(\Lambda t)^2 + \ldots\right)V^{-1}$$

$$= V\left(\begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix} + \begin{bmatrix} \lambda_1 t & & & 0 \\ & \lambda_2 t & & \\ & & \ddots & \\ 0 & & & \lambda_n t \end{bmatrix} + \begin{bmatrix} \frac{1}{2}\lambda_1^2 t^2 & & & 0 \\ & \frac{1}{2}\lambda_2^2 t^2 & & \\ & & \ddots & \\ 0 & & & \frac{1}{2}\lambda_n^2 t_n^2 \end{bmatrix} + \ldots\right)V^{-1}$$

$$= V\left(\begin{bmatrix} \exp(\lambda_1 t) & & & 0 \\ & \exp(\lambda_2 t) & & \\ & & \ddots & \\ 0 & & & \exp(\lambda_n t) \end{bmatrix}\right)V^{-1}$$

$$\boxed{\underline{\Delta x} = V\left(\begin{bmatrix} \exp(\lambda_1 t) & & & 0 \\ & \exp(\lambda_2 t) & & \\ & & \ddots & \\ 0 & & & \exp(\lambda_n t) \end{bmatrix}\right)V^{-1}\underline{\Delta x}_0}$$

$$\boxed{\underline{x} = \underline{x}_e + V\left(\begin{bmatrix} \exp(\lambda_1 t) & & & 0 \\ & \exp(\lambda_2 t) & & \\ & & \ddots & \\ 0 & & & \exp(\lambda_n t) \end{bmatrix}\right)V^{-1}\underline{\Delta x}_0}$$

$\mathrm{Re}\{\lambda\} < 0$ for all λ: stable

$\mathrm{Re}\{\lambda\} > 0$ for 1 λ: unstable

Ex Spring-mass-damper

$$m\ddot{x} = -kx - D\dot{x}$$

$$\ddot{x} = -\left(\frac{k}{m}\right)x - \left(\frac{D}{m}\right)\dot{x}$$

$$\frac{dx_1}{dt} = x_2$$

$$\frac{dx_2}{dt} = -\left(\frac{k}{m}\right)x_1 - \left(\frac{D}{m}\right)x_2$$

$$\boxed{\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{D}{m} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}$$

$$\det(A - \lambda I) = 0 \Rightarrow \begin{vmatrix} -\lambda & 1 \\ -\frac{k}{m} & -\frac{D}{m} - \lambda \end{vmatrix} = 0$$

$$-\lambda\left(-\frac{D}{m} - \lambda\right) - \left(-\frac{k}{m}\right) = 0$$

$$\lambda^2 + \frac{D}{m}\lambda + \frac{k}{m} = 0$$

$$\lambda = \frac{-\frac{D}{m} \pm \sqrt{\left(\frac{D}{m}\right)^2 - 4\frac{k}{m}}}{2}$$

Define: $\omega_n = \sqrt{\frac{k}{m}}$ $\qquad 2\zeta\omega_n = \frac{D}{m}$

$$\lambda = \frac{-2\zeta\omega_n \pm \sqrt{4\zeta^2\omega_n^2 - 4\omega_n^2}}{2} \Rightarrow \lambda = \left(-\zeta \pm \sqrt{\zeta^2 - 1}\right)\omega_n$$

<u>Case 1:</u> $\zeta = 0$ (No damper) $\Rightarrow$ $\boxed{\lambda = \pm j\omega_n \text{ Marginally Stable}}$

<u>Case 2:</u> $0 < \zeta < 1$ $\Rightarrow$ $\boxed{\lambda = \left(-\zeta \pm j\sqrt{1 - \zeta^2}\right)\omega_n \text{ Stable}}$

<u>Case 3:</u> $\zeta \geq 1$ $\Rightarrow$ $\boxed{\lambda = \left(-\zeta \pm \sqrt{\zeta^2 - 1}\right)\omega_n \text{ Stable}}$

Ex] Pendulum

$$\ddot{\theta} = -\frac{g}{l}\sin(\theta)$$

$\theta_e = 0, \pi$ from before

$\theta_e = 0$:

$$\Delta\ddot{\theta} = -\frac{g}{l}\Delta\theta$$

$$\frac{d}{dt}\begin{bmatrix}\Delta x_1 \\ \Delta x_2\end{bmatrix} = \begin{bmatrix}0 & 1 \\ -\frac{g}{l} & 0\end{bmatrix}\begin{bmatrix}\Delta x_1 \\ \Delta x_2\end{bmatrix}$$

$$\det(A-\lambda I)=0 \Rightarrow \begin{vmatrix}-\lambda & 1 \\ -\frac{g}{l} & -\lambda\end{vmatrix} = 0$$

$$\lambda^2 - \left(-\frac{g}{l}\right) = 0$$

$$\lambda^2 + \frac{g}{l} = 0$$

$$\boxed{\begin{array}{c}\lambda = \pm j\sqrt{\frac{g}{l}} \\ \text{Marginally Stable}\end{array}}$$

$\theta_e = \pi$:

$$\Delta\ddot{\theta} = \frac{g}{l}\Delta\theta$$

$$\frac{d}{dt}\begin{bmatrix}x_1 \\ x_2\end{bmatrix} = \begin{bmatrix}0 & 1 \\ \frac{g}{l} & 0\end{bmatrix}\begin{bmatrix}\Delta x_1 \\ \Delta x_2\end{bmatrix}$$

$$\det(A-\lambda I)=0 \Rightarrow \begin{vmatrix}-\lambda & 1 \\ \frac{g}{l} & -\lambda\end{vmatrix} = 0$$

$$\lambda^2 - \frac{g}{l} = 0$$

$$\boxed{\begin{array}{c}\lambda = \pm\sqrt{\frac{g}{l}} \\ \text{unstable}\end{array}}$$

Ex) Predator - Prey

$\frac{dx}{dt} = x(1-y)$ $\frac{dy}{dt} = y(x-1)$

$x_e = 1$

$y_e = 1$

$\frac{d\Delta x}{dt} = -\Delta y$ $\frac{d\Delta y}{dt} = \Delta x \Rightarrow \frac{d}{dt}\begin{bmatrix}\Delta x \\ \Delta y\end{bmatrix} = \begin{bmatrix}0 & -1 \\ 1 & 0\end{bmatrix}\begin{bmatrix}\Delta x \\ \Delta y\end{bmatrix}$

$\det(A-\lambda I) = 0 \Rightarrow \begin{vmatrix}-\lambda & -1 \\ 1 & -\lambda\end{vmatrix} = 0$

$\lambda^2 - (-1) = 0$

$\lambda^2 + 1 = 0$

$\boxed{\begin{array}{l}\lambda = \pm j \\ \text{Marginally Stable}\end{array}}$